MATHEMATICS

A NOTE ON THE ABSOLUTE BOUND FOR SYSTEMS OF LINES

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ABSTRACT

The absolute bound for the cardinality of sets of lines in $\mathbb{R}^n$ or $\mathbb{C}^n$ having a prescribed number of angles, which was first obtained by Delsarte, Goethals & Seidel, is derived in a new, very quick way.

One of the main results in DELSARTE, GOETHALS & SEIDEL [1] is the derivation of an absolute bound for the cardinality of sets of lines having a prescribed number of angles, both in real and in complex Euclidean n-space. In order to formulate this result (cf. Theorem 1) we first introduce some notation.

Let $(\cdot\,,\,\cdot)$ denote the real or Hermitian inner product in $\mathbb{R}^n$ or $\mathbb{C}^n$, respectively. Let Ω_n be the unit sphere in $\mathbb{R}^n$ or $\mathbb{C}^n$. Let $x_1, x_2, \ldots, x_n$ be real (or complex) coordinates on $\mathbb{R}^n$ (or $\mathbb{C}^n$). In the case $\mathbb{R}^n$, let Hom (l) be the class of functions on Ω_n which are restrictions to Ω_n of homogeneous polynomials of degree l in $x_1, x_2, \ldots, x_n$. In the case $\mathbb{C}^n$, let Hom (l, k) be the class of functions on Ω_n which are restrictions to Ω_n of polynomials $F(x_1, \ldots, x_n; \bar{x}_1, \ldots, \bar{x}_n)$, homogeneous of degree l in $x_1, \ldots, x_n$ and homogeneous of degree k in $\bar{x}_1, \ldots, \bar{x}_n$ (cf. KOORNWINDER [2, § 2]). Then

$$M(l) = \dim \operatorname{Hom}(l) = \binom{n+l-1}{n-1}, \tag{1}$$

$$M(l, k) = \dim \operatorname{Hom}(l, k) = \binom{n+l-1}{n-1}\binom{n+k-1}{n-1}. \tag{2}$$

Let $A = \{\alpha_1, \alpha_2, \ldots, \alpha_s\}$ be a subset of the interval $[0, 1)$. Put $\varepsilon = 1$ if $0 \in A$ and $\varepsilon = 0$ if $0 \notin A$. Let X be a finite subset of Ω_n of cardinality $|X|$ such that $|(\xi, \eta)|^2 \in A$ if $\xi, \eta \in X$ and $\xi \neq \eta$.

THEOREM 1. *$|X| \leqslant M(2s-\varepsilon)$ in the case $\mathbb{R}^n$, and $|X| \leqslant M(s, s-\varepsilon)$ in the case $\mathbb{C}^n$.*

This theorem was proved in [1, Theorem 6.1] by using the decomposition of functions on Ω_n in terms of real or complex spherical harmonics and by expressing the zonal spherical harmonics on Ω_n as Jacobi polynomials. Below we give a new, almost trivial proof of Theorem 1.

PROOF OF THEOREM 1. For each $\xi \in X$ define the polynomial

$$F_\xi(x) = ((x, \xi))^\varepsilon \prod_{\alpha \in A/\{0\}} \frac{|(x, \xi)|^2 - \alpha(x, x)}{1 - \alpha}. \tag{3}$$

Then the restriction of F_ξ to Ω_n belongs to Hom $(2s-\varepsilon)$ in the case $\mathbb{R}^n$ and to Hom $(s, s-\varepsilon)$ in the case $\mathbb{C}^n$. Furthermore

$$F_\xi(\eta) = \delta(\xi, \eta), \quad \xi, \eta \in X.$$

Hence the functions F_ξ, $\xi \in X$, are linearly independent. It follows that $|X|$ is bounded by the dimension of Hom $(2s-\varepsilon)$ or Hom $(s, s-\varepsilon)$. □

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REFERENCES

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2. KOORNWINDER, T. H., The addition formula for Jacobi polynomials, II, The Laplace type integral representation and the product formula, Math. Centrum, Amsterdam, Report TW 133 (1972).